Essential Spectrum

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The purpose of this note is to prove several equivalent characterizations of the essential spectrum.

Definition 1.1. Let $T : H \to H$ be a bounded self-adjoint operator on a Hilbert space. Then $\lambda \in \sigma_{\text{ess}}(T)$ if $T - \lambda$ is not Fredholm.

We have the elementary consequences:

Proposition 1.2. *$\sigma_{ess}(T)$ satisfies:*

(i) $\sigma_{ess}(T) \subseteq \sigma(T)$*;*

(ii) $\sigma_{ess}(T)$ *is closed;*

(iii) if K is a (self-adjoint) compact operator, then $\sigma_{ess}(T) = \sigma_{ess}(T + K)$*.*

Proof. (i) is clear since any invertible operator is Fredholm. (ii) follows from the fact that the set of Fredholm operators is open, and that the map $\lambda \mapsto T - \lambda$ is continuous. (iii) is clear since $T - \lambda$ is Fredholm implies that $T + K - \lambda$ is Fredholm. □

Now we prove the following:

Theorem 1.3. *The following are equivalent:*

(i) $\lambda \in \sigma_{ess}(T)$*;*

(ii) (Weyl's Criterion) there exists a sequence ψ_k with $\|\psi_k\| = 1$ such that

$$(T - \lambda)\psi_k \to 0$$

and ψ_k has no convergent subsequence;

(iii) λ is an eigenvalue of infinite multiplicity (i.e. $\dim \ker T - \lambda = \infty$) or there exists $\mu_n \in \sigma(T)$ such that $\mu_n \to \lambda$;

(iv) for any self-adjoint compact operator K, $\lambda \in \sigma(T + K)$;

Proof. First observe that by self-adjointness, $\ker T - \lambda = \ker(T - \lambda)^{\perp} = \overline{\operatorname{im} T - \lambda}$. In particular $T - \lambda$ is Fredholm if and only if $\dim \ker T - \lambda < \infty$ and $\operatorname{im} T - \lambda$ is closed. We show that (i) is equivalent to each of (ii), (iii), (iv).

(ii) Suppose (ii) did not hold for λ. Then there exists some $c > 0$ such that

$$\inf_{\|\psi\|=1, \psi \in \ker(T-\lambda)^\perp} \|(T-\lambda)\psi\| \geq c.$$

Indeed, if not then there would be a sequence ψ_k with $\|\psi_k\| = 1$ and $\psi_k \in \ker(T-\lambda)^\perp$ with $(T-\lambda)\psi_k \to 0$, and so since (ii) does not hold, a convergent subsequence $\psi_{n_k} \to \psi$. Then $(T-\lambda)\psi = 0$, $\|\psi\| = 1$ and $\psi \in \ker(T-\lambda)^\perp$, a contradiction. Suppose $y \in \overline{\operatorname{im} T - \lambda}$. Then there is a sequence ψ_k with $(T-\lambda)\psi_k \to \varphi$. Projecting onto $\ker(T-\lambda)^\perp$, we may assume that $\psi_k \in \ker(T-\lambda)^\perp$. In particular $(T-\lambda)\psi_k$ is Cauchy, and since

$$c\|\psi_k - \psi_j\| \leq \|(T-\lambda)(\psi_k - \psi_j)\|,$$

so is ψ_k. In particular, $\psi_k \to \psi$, and $\varphi = (T-\lambda)\psi \in \operatorname{im}(T-\lambda)$. Thus $\operatorname{im} T - \lambda$ is closed. Since (ii) does not hold, it is clear that $\dim\ker(T-\lambda) < \infty$. Thus $\lambda \notin \sigma_{\text{ess}}(T)$.

Now suppose $\lambda \notin \sigma_{\text{ess}}(T)$. Then $T - \lambda$ is Fredholm. In particular, $\operatorname{im} T - \lambda$ is closed, and so

$$(T-\lambda) : \ker(T-\lambda)^\perp \to \operatorname{im} T - \lambda = \ker(T-\lambda)^\perp$$

is invertible. Let ψ_k be any sequence with $\|\psi_k\| = 1$ and $(T-\lambda)\psi_k \to 0$. Let ψ_k' be the projection of ψ_k onto $\ker(T-\lambda)^\perp$. Then $(T-\lambda)\psi_k' \to 0$, and so by invertibility, $\psi_k' \to 0$, too. Let $\psi_k'' = \psi_k - \psi_k'$ be the projection onto $\ker T - \lambda$. Since this is a finite-diemensional space, there is a convergent subsequence ψ_{n_k}''. Thus ψ_{n_k} is convergent, so (ii) does not hold.

(iii) Suppose (iii) did not hold. Then $\dim\ker(T-\lambda) < \infty$, and either $\lambda \notin \sigma(T)$ or it is an isolated point of $\sigma(T)$. In the first case, certainly $\lambda \notin \sigma_{\text{ess}}(T)$. In the second case, $\lambda \in \mathbf{R}$ and we claim that there exists $\varepsilon > 0$ such that

$$\|(T-\lambda)\varphi\|^2 \geq \varepsilon^2 \left(\|\varphi^2\| - \|\varphi'^2\|\right),$$

where φ' is the projection of φ onto $\ker T - \lambda$. Indeed, by the Spectral Theorem,

$$\|(T-\lambda)\varphi\|^2 = \langle (T-\lambda)^2\varphi, \rangle = \int_{\sigma(T)} (\mu - \lambda)^2 \, d\langle E_\mu \varphi, \varphi\rangle.$$

Here dE_λ is the spectral measure and $d\langle E_\lambda\varphi, \varphi\rangle$ is the non-negative measure which it induces. In particular

$$\int_{\sigma(T)} 1 \, d\langle E_\mu\varphi, \varphi\rangle = \|\varphi\|^2$$

and

$$\int_{\{\lambda\}} 1 \, d\langle E_\mu\varphi, \varphi\rangle$$

is the norm of the projection of φ onto $\ker(T-\lambda)$. Since λ is isolated, there is some $\varepsilon > 0$ such that

$$(\lambda - \varepsilon, \lambda + \varepsilon) \cap \sigma(T) = \emptyset.$$

Thus, since $(\mu-\lambda)^2$ vanishes at $\mu=\lambda$.

$$\begin{aligned}
&\int_{\sigma(T)} (\mu-\lambda)^2 \; d\langle E_\mu \varphi, \varphi\rangle \\
&= \int_{\sigma(T)\setminus\{\lambda\}} (\mu-\lambda)^2 \; d\langle E_\mu \varphi, \varphi\rangle \\
&= \int_{(\lambda-\varepsilon,\lambda+\varepsilon)^c\cap\sigma(T)} (\mu-\lambda)^2 \; d\langle E_\mu \varphi, \varphi\rangle \\
&\geq \varepsilon^2 \int_{(\lambda-\varepsilon,\lambda+\varepsilon)^c\cap\sigma(T)} d\langle E_\mu \varphi, \varphi\rangle \\
&= \varepsilon^2 \int_{\sigma(T)} 1 \; d\langle E_\mu \varphi, \varphi\rangle - \varepsilon^2 \int_{\{\lambda\}} 1 \; d\langle E_\mu \varphi, \varphi\rangle.
\end{aligned}$$

Putting it all together yields that

$$\|(T-\lambda)\varphi\|^2 \geq \varepsilon^2 \left(\|\varphi^2\| - \|\varphi'^2\| \right).$$

In particular $(T-\lambda)$ satisfies the estimate

$$\inf_{\|\psi\|=1, \psi\in\ker(T-\lambda)^\perp} \|(T-\lambda)\psi\| \geq \varepsilon.$$

As above, this implies that $\operatorname{im}(T-\lambda)$ is closed. Thus, in all $T-\lambda$ is Fredholm.

Now suppose $\lambda \notin \sigma_{\text{ess}}(T)$. Then $T-\lambda$ is Fredholm, and so $\dim(\ker(T-\lambda)) < \infty$. If $\lambda \in \mathbf{C}$, then λ is a positive distance away from the spectrum. These together means that (iii) does not hold. We now handle $\lambda \in \mathbf{R}$.

Set $S = T-\lambda$. If $\lambda \in \mathbf{R}$, Then S is self-adjoint and Fredholm. We may picture S as a 2×2 matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & S_{\operatorname{im} S} \end{pmatrix},$$

where we use the decomposition

$$H = \ker S + \operatorname{im} S = \ker S + \ker S^\perp$$

to make sense of the matrix. Since $\ker S = \operatorname{im} S^\perp$ by self-adjointness, $S-\mu$ looks like the matrix

$$\begin{pmatrix} -\mu & 0 \\ 0 & S_{\operatorname{im} S} - \mu \end{pmatrix}.$$

Since $S|_{\operatorname{im} S} \to \operatorname{im} S$ is invertible, $S_{\operatorname{im} S} - \mu$ is invertible for small μ, it is clear from looking at the matrix that this means that so is $S-\mu$. Thus $T-\lambda-\mu$ is invertible for μ small, i.e. λ, should it be in $\sigma(T)$, is an isolated point. This completes the proof that (iii) does not hold.

(iv) Suppose $\lambda \in \sigma_{\text{ess}}(T)$. Then by the proposition, for any self-adjoint compact operator K, $\lambda \in \sigma_{\text{ess}}(T+K) \subseteq \sigma(T+K)$. Conversely, suppose $\lambda \notin \sigma_{\text{ess}}(T)$. If $\lambda \in \mathbf{C}$, then for any self-adjoint compact K, $\lambda \notin \sigma(T+K)$. If $\lambda \in \mathbf{R}$, write $S = T - \lambda$ as above, and let P be the orthogonal projection onto $\ker S$. Then P has finite rank and is compact (and is self-adjoint since it is a projection). $S + P$ is invertible, since as a matrix it looks like

$$\begin{pmatrix} 1 & 0 \\ 0 & S_{\operatorname{im} S} \end{pmatrix}.$$

Therefore, $\lambda \notin \sigma(T+P)$, and so (iv) does not hold.

□

Remark 1.4. The set $\sigma(T) \setminus \sigma_{\text{ess}}(T)$ is often call the discrete spectrum, denoted $\sigma_{\text{discr}}(T)$, and by (iii) is characterized by the property that $\lambda \in \sigma_{\text{discr}}(T)$ if and only if $0 < \dim\ker T - \lambda < \infty$ and λ is an isolated point in the spectrum. Indeed, the only thing which needs justification is the strict inequality $\dim\ker T - \lambda > 0$. But $T - \lambda$ is Fredholm, and so $\operatorname{im} T - \lambda$ is closed, so the only way $T - \lambda$ can fail to be invertible is if $T - \lambda$ has non-trivial kernel.

Remark 1.5. We remark that Weyl's criterion has an analogue for $\sigma(T)$: $\lambda \in \sigma(T)$ if and only if there exists a sequence ψ_k with $\|\psi_k\| = 1$ and $(T-\lambda)\psi_k \to 0$. There is no assumption on not having a convergent subsequence. The proof is similar to (ii).